A polynomial time algorithm for submodular 4-partition

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January 23, 2023

UESTC

Submodular k-partition

- A function $f : 2^V \to \mathbb{R}$ is submodular if $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq V$.

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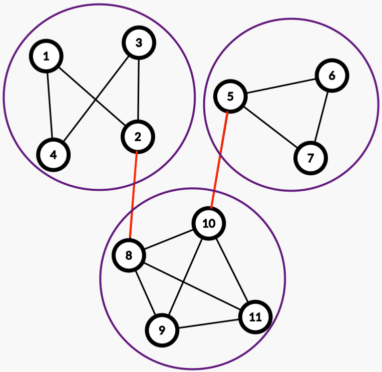
Goal: A polynomial time algorithm for **fixed** k.

Examples: Min k-cut in graphs

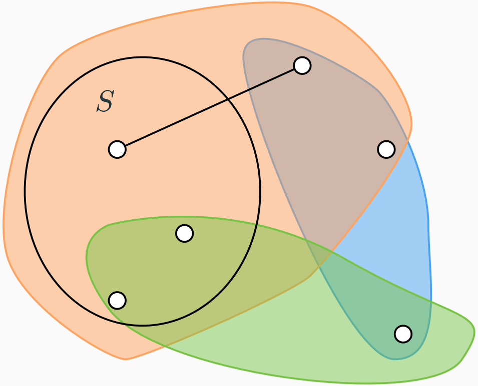


$G = (V, E)$ is a graph.

- F is a k-cut if $G - F$ has at least k components.
- c is the cut function, $c(S)$ is number of edges with 1 vertex in S and one outside of S.

  c is submodular.
- **min-k-cut = min submodular k-partition of c.**

Examples: Min k-cut in hypergraphs



- min k-partition: min submodular k-partition of the cut function c.
- min k-cut: same definition as graph minimum k-cut.
- hypergraph k-partition $\neq$ hypergraph k-cut!
- Let $h(e) \in e$ be a designated vertex of e. $f(S)$ is number of edges with $h(e) \in S$ and some vertex of e outside of S.
- **min-k-cut = min submodular k-partition of f.**

Previous works on GRAPH k-cut

- Fix a partition class: $n^{\Theta(k^2)}$ [Goldschmidt-Hochbaum 94].
- Randomized contraction: $\tilde{O}(n^{2(k-1)})$ [Karger-Stein 96].
- Divide and conquer: $O(n^{(4+o(1))k})$ [Kamidoi-Yoshida-Nagamochi 07], $O(n^{(4-o(1))k})$ [Xiao 08].
- Tree packing: $\tilde{O}(n^{2k})$ [Thorup 08], $\tilde{O}(n^{2k-1})$ [Chekuri-Quanrud-X 20]
- Optimum randomized contraction $\tilde{O}(n^k)$ [Gupta-Harris-Lee-Li 20].

Previous works on HYPERGRAPH k-cut

Polynomial time algorithms:

- $k = 2$
  - Vertex ordering: [Klimmek-Wagner 96, Queyranne 98, Mak-Wong 00].
  - Randomized contraction: [Ghaffari-Karger-Panigrahi 17].
- $k = 3$: Deterministic contraction [Xiao 08].
- Constant rank: Hypertree packing [Fukunaga 10].
- General k
  - randomized algorithm [Chandrasekaran-X-Yu 18, Fox-Panigrahi-Zhang 19].
  - deterministic algorithm [Chekuri-Chandrasekaran 20]

Previous works on Submodular k-partition

- $k = 2$: Reduces to symmetric submodular minimization. i.e. $g(S) = f(S) + f(V \setminus S)$.
- $k = 3$: Generalizes hypergraph 3-cut algorithm [Okumoto-Fukunaga-Nagamochi 10]
- Open for $k \geq 4$.

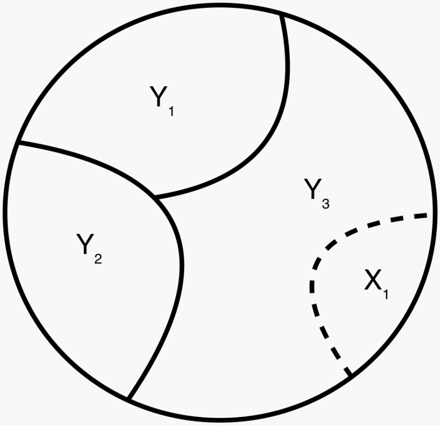
Our Result

$\tau(n)$ time to minimize a submodular function on n vertices.

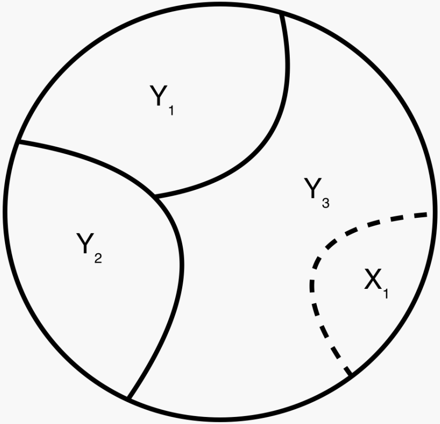
Theorem
There exists an $O(n^6\tau(n))$ time algorithm for submodular 4-partition.

Generalizes the deterministic contraction approach for submodular 3-partition.

Warmup: Submodular 3-partition



Definition (Noncrossing)
A partition $\mathcal{X}$ *is noncrossing with* a partition $\mathcal{Y}$ if there is a component of $\mathcal{X}$ that is contained in some component of $\mathcal{Y}$.



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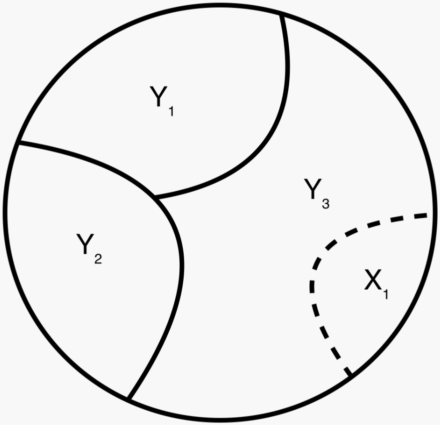
Every min-2-partition is noncrossing with some min-3-partition.



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Simple case analysis.

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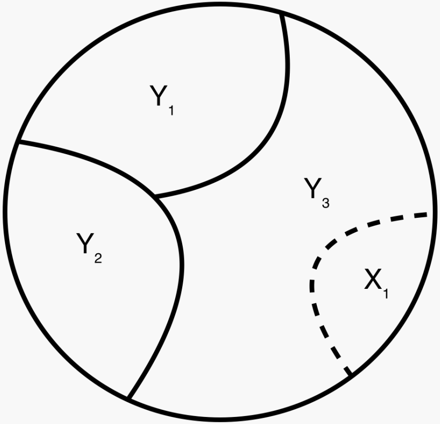

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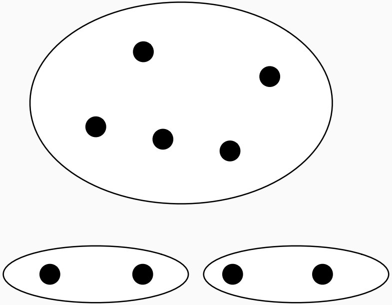
Every min-2-partition is noncrossing with some min-3-partition.

Simple case analysis. Hints a contraction algorithm.

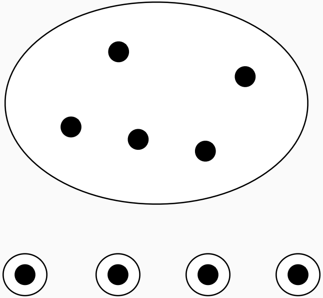
Warmup: Submodular 3-partition

A partition is called *h*-**size** if all its components contain at least *h* elements.



2-size

A partition is called **non-trivial** if at least two partition classes has size at least 2.



Trivial

Combinatorial Structure

Theorem

Let f be a submodular function on at least 7 vertices. If all minimum 3-partition are 2-size, then every minimum non-trivial 2-partition is noncrossing with some minimum 3-partition.

Combinatorial Structure

Theorem

Let f be a submodular function on at least 7 vertices. If all minimum 3-partition are 2-size, then every minimum non-trivial 2-partition is noncrossing with some minimum 3-partition.

$\underline{\textsc{Min3Partition}(f):}$

- $V \leftarrow \mathit{domain}(f)$
- if $|V| \leq 6$
  - return the optimum by brute force
- for $X \in \binom{V}{1}$
  - add candidate $\{X\} \cup \textsc{Min2Partition}(f_{\setminus X})$
- $\mathcal{X} \leftarrow \textsc{MinNonTrivial2Partition}(f)$
- for $X \in \mathcal{X}$
  - add candidate $\textsc{Min3Partition}(f_{/X})$
- return minimum over all candidates

Running Time Analysis

```
MIN3PARTITION(f):
    V ← domain(f)
    if |V| ≤ 6
        return the optimum by brute force
    for X ∈ (V choose 1)
        add candidate {X} ∪ MIN2PARTITION(f_{\X})
    𝒳 ← MINNONTRIVIAL2PARTITION(f)
    for X ∈ 𝒳
        add candidate MIN3PARTITION(f_{/X})
    return minimum over all candidates
```

$$T(n) = \max_{\substack{a+b=n \\ 1 \le a \le b \le n-2}} T(a+1) + T(b+1) + O(n^c) = O(n^{c+1}).$$

Theorem
Every min-3-partition is noncrossing with a min-4-partition.

Submodular 4-partition

Theorem
Every min-3-partition is noncrossing with a min-4-partition.

Noncrossing is insufficient for polynomial time algorithm for 4-partitions.

Let $\mathcal{X} = \{X_1, X_2, X_3\}$ such that $|X_1| = n - 4$, $|X_2|, |X_3| = 2$.

Same algorithm gives us running time.

$$T(n) \geq 2T(n-1) + O(n^c)$$

$T(n)$ **is exponential!**

Compatible

Definition (Noncrossing)
A partition $\mathcal{X}$ *is noncrossing with* a partition $\mathcal{Y}$ if there is 1 component of $\mathcal{X}$ that is contained in some component of $\mathcal{Y}$.

Compatible

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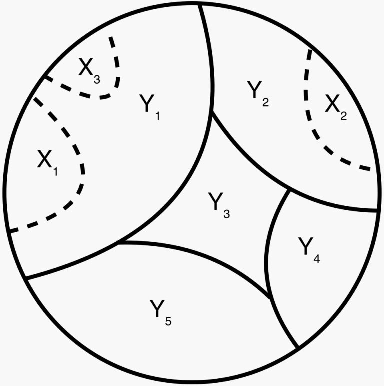
A partition $\mathcal{X}$ *is noncrossing with* a partition $\mathcal{Y}$ if there is 1 component of $\mathcal{X}$ that is contained in some component of $\mathcal{Y}$.

Definition (Compatible)
A partition $\mathcal{X}$ is *compatible* with partition $\mathcal{Y}$, if there are $|\mathcal{X}| - 1$ components of $\mathcal{X}$ that each is contained inside some component of $\mathcal{Y}$.

Noncrossing $=$ Compatible for 2-partitions

Compatible



Compatible allows us to contract $|\mathcal{X}| - 1$ sets at the same time.

Non-trivial make sure each contraction of $|\mathcal{X}| - 1$ sets decreases the number of vertices.

$$T(n) = \max_{\substack{\sum_{i=1}^{k} a_i = n \\ 1 \leq a_i \leq n-k}} \sum_{i=1}^{k} T(a_i + k - 1) + O(n^c) = O(n^{c+1}).$$

Combinatorial Structure

Theorem (Compatibility of 2-partition and 3-partition)
Let f be a submodular function on at least $(2 \times 3) + 1$ vertices. If all minimum 3-partition are 2-size, then every minimum non-trivial 2-partition is compatible with some minimum 3-partition.

Combinatorial Structure

Theorem (Compatibility of 2-partition and 3-partition)
Let f be a submodular function on at least $(2 \times 3) + 1$ vertices. If all minimum 3-partition are 2-size, then every minimum non-trivial 2-partition is compatible with some minimum 3-partition.

Theorem (Compatibility of 3-partition and 4-partition)
Let f be a submodular function on at least $(3 \times 4) + 1$ vertices. If all minimum 4-partition are 3-size, then every minimum non-trivial 3-partition is compatible with some minimum 4-partition.

Proof.
Case Analysis. Lot of cases. □

Algorithm for submodular 4-partition

```
Min4Partition(f):
    V ← domain(f)
    if |V| ≤ 12
        return the optimum by brute force
    for X ∈ (V choose 1) ∪ (V choose 2)
        add candidate {X} ∪ Min3Partition(f_{\X})
    𝒳 ← MinNonTrivial3Partition(f)
    for {A, B} ∈ (𝒳 choose 2)
        add candidate Min4Partition((f_{/A})_{/B})
    return minimum over all candidates
```

$\textsc{Min4Partition}(f)$:

- $V \leftarrow \mathit{domain}(f)$
- if $|V| \leq 12$
  - return the optimum by brute force
- for $X \in \binom{V}{1} \bigcup \binom{V}{2}$
  - add candidate $\{X\} \cup \textsc{Min3Partition}(f_{\setminus X})$
- $\mathcal{X} \leftarrow \textsc{MinNonTrivial3Partition}(f)$
- for $\{A, B\} \in \binom{\mathcal{X}}{2}$
  - add candidate $\textsc{Min4Partition}((f_{/A})_{/B})$
- return minimum over all candidates

Find a minimum non-trivial 3-partition is in P.

Compatibility for larger k?

Conjecture
Every minimum $k-1$-partition is compatible with some minimum k-partition.

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Every minimum $k-1$-partition is compatible with some minimum k-partition.

FALSE! Counterexample in graphs for $k = 5$!

Open problems

- **Algorithmic**: Polynomial time submodular k-partition algorithm for $k \geq 5$?
- **Combinatorial**:
  - Every min $k-1$-partition is **noncrossing** with a min k-partition? (it is true for $k = 5$!)
  - Every min $k-1$-partition has at least t_k parts that each is a subset of some part in a min k-partition, how large can t_k be?

Thank You!